APPLICATION OF THE BOUNDARY-ELEMENT METHOD TO CALCULATE SUBSONIC FLOW AROUND A PROFILE

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The broad capabilities of computers have led to the development of various numerical methods to calculate profile characteristics. Most of these methods use a finite-difference representation of the defining partial differential equations. The method of discrete distribution of singularities for calculating incompressible potential flow is fundamentally different [1]. It has been shown [2] that this method can also be extended to solving two-dimensional Poisson equations, with singularities distributed not only on the boundary, but within the flow field. Thus it makes it possible to examine transonic shock-free flow around profiles.

The boundary-element method allows problems to be reduced to one-dimensional ones and to reduce computer time significantly. It is based on the assumption that the density quantities which enter into the integrals are constant in small cells and in every small element of the boundary. Because the boundary-element method automatically satisfies allowable boundary conditions at infinity, only boundaries such as the contour of the body profile have to be discretized. The discretization region does not increase the order of the final system of algebraic finite-difference equations, including the Kutta–Zhukovskii conditions. The formulation of the problem which leads to its solution is the boundary integral equation itself. The error introduced, if the numerical integration is done on a curvilinear boundary, can be made very small. Also, numerical integration is always more stable in an exact process than numerical differentiation.

If we use the relationships

$$\operatorname{div} \mathbf{v} = \frac{\partial v}{\partial s} + v\frac{\partial \theta}{\partial n}, \quad \frac{\partial v}{\partial s} = \frac{v}{M^2 - 1}\frac{\partial \theta}{\partial n}, \quad \frac{\partial v}{\partial n} = v\frac{\partial \theta}{\partial s} \tag{1}$$

the differential equations for vortex-free nonviscous potential flow can be written in the form

$$\nabla \mathbf{v} = M^2 \partial v/\partial s. \tag{2}$$

Here s is the direction tangent to the flow line; n is perpendicular to s; θ is the inclination angle of the velocity vector; and M is the local Mach number. Because the flow is vortex free, after we introduce a perturbation potential φ such that $\mathbf{v} = \nabla\Phi + \nabla\varphi$, we have from (1) and (2) that

$$\Delta\varphi = M^2\frac{\partial v}{\partial s} = Q(M, v), \tag{3}$$

where $\mathbf{v}_0 = \nabla\Phi$ is the velocity of the unperturbed flow. The impermeability on the boundary contour requires that φ satisfies

$$\varphi'_n = -\mathbf{v}_0 \cdot \mathbf{n}. \tag{4}$$

Numerical realization of the Kutta–Zhukovskii condition with a zero angle on the trailing edge requires that the tangential velocities at the points ξ^+ and ξ^- be equal on the upper and lower surfaces of the profile, respectively, if the condition

$$h_t = |\xi^+ - \xi_k| = |\xi^- - \xi_k| \sim \varepsilon \tag{5}$$

is fulfilled ($\varepsilon \ll 1$, and ξ_k corresponds to the end point of the profile).

By using finite differences for the derivatives, we represent (5) in the form

$$\varphi(\xi^+) - \varphi(\xi^-) = h_t\,[\mathbf{v}_0\cdot\mathbf{t}(\xi^+) - \mathbf{v}_0\cdot\mathbf{t}(\xi^-)], \tag{6}$$

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where t is a vector tangent to the profile at point ξ. By using Green's theorem, the potential φ can be written as

$$\varphi(\xi) = \int_C [\varphi(x) F(x, \xi) - v_n(x) G(x, \xi)]\, dl + \int_\Omega \dot{Q}(x) G(x, \xi)\, d\Omega,$$
$$G(x, \xi) = -\frac{1}{2\pi} \ln r,\ r = \sqrt{(x_1(x) - x_1(\xi))^2 + (x_2(x) - x_2(\xi))^2}, \tag{7}$$
$$F(x, \xi) = n_i(x) \frac{\partial}{\partial x_i} G(x, \xi) = n_i(x)(x_i(x) - x_i(\xi))/2\pi r^2,$$

where v_n is the derivative of φ normal to the boundary.

The boundary problem is solved numerically by the method of discrete distribution of singularities; it is based on the assumption that the densities of Q, φ, and v are constant in small cells in the region Ω and on each small boundary element, which is represented as a straight line segment.

As mentioned before, the boundary-element method automatically satisfies allowable boundary conditions at infinity; therefore, only the boundary contour has to be discretized. Let it be divided into (N − 1) elements, on which the parameters are considered constant. In practice, the region in which Q differs from zero is bounded by flow lines above and below the profile and is divided into K cells. Then (7) is transformed to

$$\frac{1}{2}\varphi(\xi_0^p) = \sum_{q=1}^{N-1} \varphi(x^q) \int_{\Delta C_q} F(x^q, \xi_0^p)\, dl(x^p) - \sum_{q=1}^{N-1} v_n(x^q) \int_{\Delta C_q} G(x^q, \xi_0^p)\, dl(x^p) + \sum_{k=1}^{K} Q(x^k) \int_{\Delta\Omega_k} G(x^k, \xi_0^p)\, d\Omega(x^k), \tag{8}$$

where ξ_0^p pertains to the p-th boundary element; q is the number of the boundary element; and $v_n(x) \equiv \varphi_n'$. By starting from (8), satisfying (4) at a finite number (N − 1) of collocation points, and requiring that (6) be fulfilled at point ξ_N, we obtain a system of nonlinear equations, where the unknowns are densities φ on the boundary elements:

$$F^c \boldsymbol{\varphi} = G^c \mathbf{v}_n + G^\Omega \mathbf{Q}. \tag{9}$$

The dimensions of quantities entering into (9) are

$$\dim \|F^c\| = (N \times N), \quad \dim \|G^c\| = (N \times N),$$
$$\dim \|G^\Omega\| = (N \times K), \quad \dim \boldsymbol{\varphi} = N, \quad \dim \mathbf{v}_n = N, \quad \dim \mathbf{Q} = K.$$

Here

$$F_{qp} = \begin{cases} \overline{F}_{qp}, & q \neq p,\ q = \overline{1, N}, \\ 1/2 - \overline{F}_{qp}, & p = \overline{1, N-1}, \end{cases} \qquad \overline{F}_{qp} = \int_{\Delta C_q} F(x^q, \xi^p)\, dl_q,$$
$$G_{qp}^c = \int_{\Delta C_q} G(x^q, \xi^p)\, dl_q, \quad p = \overline{1, N-1}, \quad q = \overline{1, N};$$
$$G_{l,p}^\Omega = \int_{\Delta\Omega_{lp}} G(x, \xi^{lp})\, d\omega_{lp}, \quad l = \overline{1, K}, \quad p = \overline{1, N-1}, \quad \xi^{lp} \in \Omega.$$

The last equation in (9) corresponds to condition (6) for p = N.

In order to determine Q as a function of M and v, we use a finite-difference method with values of φ calculated in ΔQ by the formula

$$\varphi(\xi_0^p) = \sum_{q=1}^{N-1} \varphi(x^q) \int_{\Delta C_q} F(x^q, \xi_0^p)\, dl(x^p) - \sum_{q=1}^{N-1} v_n(x^q) \int_{\Delta C_q} G(x^q, \xi_0^p)\, dl(x^p) + \sum_{k=1}^{K} Q(x^k) \int_{\Delta\Omega_k} G(x^k, \xi_0^p)\, d\Omega(x^k),$$

where ξ_0^p pertains to ΔQ_p; $\varphi(x^q)$ is the value obtained in the last i-th iteration; and $Q(x^k)$ is the value from the (i − 1)-th iteration. The iteration process is limited by the formula

$$\boldsymbol{\varphi}^{(i)} = \boldsymbol{\varphi}^{(i-1)} - \lambda H(\boldsymbol{\varphi}^{(i-1)})$$
$$(H(\boldsymbol{\varphi}^{(i-1)}) = F^c \boldsymbol{\varphi}^{(i-1)} - G^c \mathbf{v}_n - G^\Omega \mathbf{Q}^{(i-1)}).$$

For its convergence, it is always possible to select appropriate values of λ under the condition that the derivative of H with respect to $\boldsymbol{\varphi}$ is limited by the constant K_0 [3]. This is obvious for the first two terms. The derivatives of Q with respect to $\boldsymbol{\varphi}$ are bounded for any triangulation of the region, because the nonlinear part can be represented in the form

$$Q = \frac{(v/v_0)^2 M_0^2}{1 - \frac{\gamma - 1}{2} M_0^2 [1 - (v/v_0)^2]} \frac{\partial v}{\partial s},$$

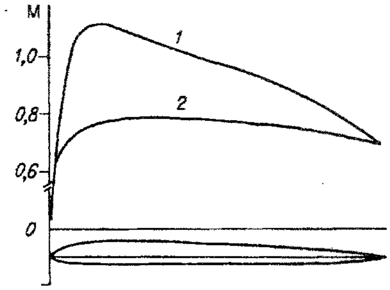

Fig. 1

and v and $\partial v/\partial s$, which enter into this expression, are computed n terms of finite differences of $\varphi(\xi_0^0)$. The process converges in six to eight steps with a good initial approximation to φ_0; here φ_0 corresponds to the solution of Eq. (3) with a zero right side, i.e., an incompressible fluid. The calculation time is ~20 sec on a PC/AT 386. All this is also valid for the transonic region, but only until a shock forms which leads to the growth of K_0 in the region and to a deterioration and then absence of convergence.

Integrals containing $\ln r$ are "weakly singular" and are calculated in the usual manner along the boundary element which passes through the singular point $x^i = \xi^i$, but this function has no singularity after integration. Integrals with $F(x, \xi)$ which contain a singularity on the order of $1/r$ are "singular." These integrals assume that matrix of the system of algebraic equations is mainly diagonal and therefore assure the stability of the solution at each integration [4].

As an example, we chose a profile for transonic shock-free flight. Figure 1 shows the distribution of the Mach number for the conditions $M_0 = 0.75$ and $\alpha = 0$ (curves 1 and 2 correspond to the upper and lower contours of the profile). The experimental and calculated integral profile characteristics are

$$C_y^e = 0.129, \quad C_y^c = 0{,}130,$$
$$m_z^e = -0.046, \quad m_z^c = -0.049.$$

LITERATURE CITED

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